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1993 J. Phys. A: Math. Gen. 26 4407

(http://iopscience.iop.org/0305-4470/26/17/045)

J. Phys. A: Math. Gen. 26 (1993) 4407–4417. Printed in the UK

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Received 1 December 1992

Abstract. In this paper we consider a linear spectral problem associated with the Levi hierarchy of equations. A Bargmann system under the so-called Bargmann constraint between the eigenfunctions and potentials is proposed, and is proved to be a completely integrable Hamiltonian system in the Liouville sense. Moreover, by the involutive solution of the compatible systems under the Bargmann constraint we present the involutive representation of solutions of the Levi hierarchy. Especially, we obtain the involutive representation of the solution of the well known Burgers equation $u_t = -u_{xx} + 2uu_x$.

1. Introduction

It is a very important topic for us to look for new finite-dimensional completely integrable system. The beautiful Liouville–Arnold theorem is the basis of the geometrical theory of finite-dimensional Hamiltonian systems [1, 2]. In the light of this theorem, the key to the complete integrability of a finite-dimensional Hamiltonian system lies in the existence of an involutive system of conserved integrals. There are some approaches to producing finite-dimensional completely integrable systems, such as the well-known isospectal technique of Lax [3] and the method of constraining infinite–dimensional integrable systems on finite-dimensional invariant manifolds [4–6].

In recent years, along with the unceasing deep study of the solution system, new vitality has been absorbed to the theory of classical integrable systems. In 1989 Cewen Cao first presented a method of the so-called nonlinearization [7] of the Lax system for soliton equations and by this effective method some new finite-dimensinal completely integrable systems in the Liouville sense have been successfully elaborated [8–16] in explicit form. Another important application of the nonlinearization method is that the solution of the soliton equation associated with an eigenvalue problem is reduced to solving the compatible system of nonlinear ordinary differential equations [17–22].

In this paper, we introduce the eigenvalue problem

$$y_x = My \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad M = \begin{pmatrix} -\frac{1}{2}\lambda + \dfrac{u-v}{2} & u \\ v & \frac{1}{2}\lambda + \dfrac{v-u}{2} \end{pmatrix} \tag{1.1}$$

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which is equivalent to the eigenvalue problem proposed by Levi [23], where u and v are two scalar potentials, λ is a constant spectral parameter, $y_x=\partial y/\partial x$. From (1.1) we derive the Levi hierarchy [23, 24] of nonlinear evolution equations and describe the Lax represenation of this hierarchy [25]. Two typical equations in the Levi hierarchy are as follows

$$\begin{aligned} u_t &= -u_{xx}+2uu_x-2(uv)_x \\ v_t &= v_{xx}-2vv_x+2(uv)_x \end{aligned} \tag{1.2}$$

$$\begin{aligned} u_t &= (u_{xx}+3u_x v-3uu_x+u^3+3uv^2-6u^2v)_x \\ v_t &= (v_{xx}+3v_x u-3vv_x+v^3+3vu^2-6v^2u)_x. \end{aligned} \tag{1.3}$$

If we let $i\lambda\to\lambda$ and $u=v^*$ in (1.1), then (1.2) becomes exactly the new nonlinear Schrödinger equation which is studied by Levi [23]. As $v=0$ and $u=v$, (1.2) and (1.3) are separately reduced to the well known Burgers equation $u_t=-u_{xx}+2uu_x$ and the well known Mkdv equation $u_t=u_{xxx}-6u^2u_x$. The Hamiltonian structure of the Levi hierarchy has been established in [24]. In the present paper, through the nonlinearization of the eigenvalue problem (1.1) (i.e. the spatial paret of the Lax pair for the Levi hierarchy) we obtain a Bargmann system under a kind of constraint (called the Bargmann constraint) between the eigenfunctions and potentials. We prove that this Bargmann system is a complete integrable Hamiltonian system in the Liouville sense and its involutive system of conserved integrals is exactly generated by the nonlinearization of the time part of the Lax pair for the Levi hierarchy. Moreover, the involutive representation of solutions of the Levi hierarchy is obtained by using the involutive solution of the compatible system under the Bargmann constraint. Especially, the involutive representation of the solution of the well known Burgers equation $u_t=-u_{xx}+2uu_x$ is obtained.

2. The Levi hierarchy of equations and their Lax representation [25]

Consider the eigenvalue problem studied by Levi [23]

$$\psi_x=U\psi \qquad \psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \qquad U=\begin{pmatrix}-u & \mu u\\ -\mu & \mu^2+v\end{pmatrix} \tag{2.1}$$

where μ is a spectral parameter, u and v are two potentials.

Proposition 2.1 Let $\lambda=\mu^2$, then (2.1) is equivalent to (1.1).

Proof. Make the transformation

$$y=\exp\left(-\tfrac{1}{2}\lambda x-\partial^{-1}\frac{v-u}{2}\right)\begin{pmatrix}1 & 0\\ -1 & \mu\end{pmatrix}\psi \tag{2.2}$$

where $\partial=\partial/\partial x$, $\partial\partial^{-1}=\partial^{-1}\partial=1$. Substituting (2.2) into (1.1), by simple calculation we know that (1.1) can be reduced to (2.1).

Equation (2.1) can become (1.1) by the inverse of (2.2)

$$\psi=\exp\left(\tfrac{1}{2}\lambda x-\partial^{-1}\frac{v-u}{2}\right)\begin{pmatrix}1 & 0\\ \frac{1}{\mu} & \frac{1}{\mu}\end{pmatrix}y \tag{2.3}$$

Because of this proposition, we study the eigenvalue problem (1.1) below.

Let y satisfy (1.1), $\nabla\lambda=(y_1y_2+y_2^2, -y_1y_2-y_1^2)^T$. Through direct calculations we have

$$K\nabla\lambda=\lambda J\nabla\lambda \tag{2.4}$$

where K and J are two skew-symmetric operators

$$K=\begin{pmatrix} -u\partial-\partial u & -\partial^2-v\partial+\partial u \\ \partial^2-\partial v+u\partial & v\partial+\partial v \end{pmatrix} \qquad J=\begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{2.5}$$

which are called the pair of Lenard's operators. In the case that λ and y are the eigenvalue and the corresponding eigenfunction of (1.1), it holds that $\nabla\lambda=(\delta\lambda/\delta u, \delta\lambda/\delta v)^T$ under periodic boundary conditions or decaying at infinity boundary conditions.

Consider the Lenard's sequence of gradient $\{G_j\}$ defined as follows

$$\begin{aligned} &KG_{j-1}=JG_j, j=0,1,2,\ldots. \\ &G_{-1}=(0,1)^T\in \operatorname{Ker} J=\{G|JG=0\}. \end{aligned} \tag{2.6}$$

It isn't difficult to determine that $G_j=(G_j^{(1)}, G_j^{(2)})^T$ can be recursively determined. The component of G_j is a polynomial of u, v and their derivatives and is unique if its constant term is required to be zero. $X_j(u,v)\triangleq JG_j$ is called the Levi vector field. The first few being

$$X_0=\begin{pmatrix} u_x \\ v_x \end{pmatrix} \qquad G_0=\begin{pmatrix} v \\ u \end{pmatrix} \qquad X_1=\begin{pmatrix} -u_{xx}+2uu_x-2(uv)_x \\ v_{xx}-2vv_x+2(uv)_x \end{pmatrix}$$

$$G_1=\begin{pmatrix} v_x+2uv-v^2 \\ -u_x-2uv+u^2 \end{pmatrix}$$

$$X_2=\begin{pmatrix} (u_{xx}+3u_xv-3uu_x+u^3+3uv^2-6u^2v)_x \\ (v_{xx}+3v_xu-3vv_x+v^3+3vu^2-6v^2u)_x \end{pmatrix}$$

$$G_2=\begin{pmatrix} v_{xx}+3v_xu-3vv_x+v^3+3vu^2-6v^2u \\ u_{xx}+3u_xv-3uu_x+u^3+3uv^2-6u^2v \end{pmatrix}.$$

The Levi hierarchy [23, 24] of nonlinear evolution equations can be produced by the Levi vector field $X_m(u,v)$ here, i.e.

$$(u,v)_t^T=X_m(u,v)=J\mathscr{L}^mG_0 \qquad m=0,1,2,\ldots \tag{2.7}$$

where

$$\mathscr{L}=J^{-1}K=\begin{pmatrix} \partial-v+\partial^{-1}u\partial & v+\partial^{-1}v\partial \\ -u-\partial^{-1}u\partial & -\partial+u-\partial^{-1}v\partial \end{pmatrix}$$

which is exactly the recursive operator in [24]. $m=1$ and $m=2$, (2.7) is (1.2) and

(1.3), respectively.

In [25], it is shown that (2.7) has the Lax representation

$$Ly=\frac{\lambda}{2}y \tag{2.8}$$

$$y_t=W_m y \tag{2.9}$$

where

$$L=\begin{pmatrix} -\partial+\frac{u-v}{2} & u \\ -v & \partial+\frac{u-v}{2} \end{pmatrix} \tag{2.10}$$

$$W_m=\sum_{j=0}^{m}\begin{pmatrix} -\frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+(G_{j-1}^{(2)}-G_{j-1}^{(1)})\partial & -G_{j-1,x}^{(2)} \\ G_{j-1,x}^{(1)} & \frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+(G_{j-1}^{(2)}-G_{j-1}^{(1)})\partial \end{pmatrix}(2L)^{m-j}. \tag{2.11}$$

The Lax representation (2.8) and (2.9) can be reduced as follows

$$y_x=My=\begin{pmatrix} -\frac{1}{2}\lambda+\frac{u-v}{2} & u \\ v & \frac{1}{2}\lambda+\frac{v-u}{2} \end{pmatrix}y \tag{1.1}$$

$$y_t=\sum_{j=0}^{m}\Bigg[\begin{pmatrix} -\frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+\frac{1}{2}(u-v)(G_{j-1}^{(2)}-G_{j-1}^{(1)}) & -G_{j-1,x}^{(2)}+u(G_{j-1}^{(2)}-G_{j-1}^{(1)}) \\ G_{j-1,x}^{(1)}+v(G_{j-1}^{(2)}-G_{j-1}^{(1)}) & \frac{1}{2}(G_{j-1}^{(1)}+G_{j-1}^{(2)})_x+\frac{1}{2}(v-u)(G_{j-1}^{(2)}-G_{j-1}^{(1)}) \end{pmatrix} \times\lambda^{m-j}y+\begin{pmatrix} \frac{1}{2}(G_{j-1}^{(1)}-G_{j-1}^{(2)}) & 0 \\ 0 & \frac{1}{2}(G_{j-1}^{(2)}-G_{j-1}^{(1)}) \end{pmatrix}\lambda^{m+1-j}y\Bigg] \tag{2.12}$$

where $G_{j-1}=(G_{j-1}^{(1)}, G_{j-1}^{(2)})^T$ $(j=0,1,2,\ldots)$ is the Lenard's sequence of gradient.

3. A Bargmann system and its complete integrability in the Liouville sense

Let λ_j ($j=1,2,\ldots,N$) and $y=(q_j,p_j)^T$ be different eigenvalues and corresponding eigenfunctions of (1.1). Consider the Bargmann constraint [8]

$$G_0=\sum_{j=1}^{N}\nabla\lambda_j$$

i.e.

$$u=-\langle p+q,q\rangle, v=\langle p+q,p\rangle \tag{3.1}$$

where $p=(p_1,\ldots p_N)^T$, $q=(q_1,\ldots q_N)^T$, $\langle\cdot,\cdot\rangle$ is the standard inner product in R^N. The nonlinearization of (1.1) (i.e. the spatial part of the Lax representation for the Levi hierarchy) under (3.1) yields a Bargmann system

$$\text{(B)}\quad \begin{cases} q_x=-\frac{1}{2}\Lambda q-\frac{1}{2}\langle p+q,p+q\rangle q-\langle p+q,q\rangle p \\ p_x=\langle p+q,p\rangle q+\frac{1}{2}\Lambda p+\frac{1}{2}\langle p+q,p+q\rangle p \end{cases} \tag{3.2}$$

where $\Lambda=\mathrm{diag}(\lambda_1,\ldots\lambda_N)$.

Proposition 3.1. The Bargmann system (3.2) can be expressed as the Hamiltonian form

$$\begin{cases} q_x=\partial H/\partial p \\ p_x=-\partial H/\partial q \end{cases} \tag{3.3}$$

with the Hamiltonian function

$$H=-\tfrac{1}{2}\langle\Lambda p,q\rangle-\tfrac{1}{2}\langle p+q,p+q\rangle\langle p,q\rangle-\frac{1}{2}\begin{vmatrix}\langle q,q\rangle & \langle p,q\rangle \\ \langle p,q\rangle & \langle p,p\rangle\end{vmatrix}. \tag{3.4}$$

Proof. Direct.

The Poisson bracket of two functions E, F in the symplectic space $(R^{2N}, dp\wedge dq)$ is defined as

$$(E,F)=\sum_{j=1}^{N}\left(\frac{\partial E}{\partial q_j}\frac{\partial F}{\partial p_j}-\frac{\partial E}{\partial p_j}\frac{\partial F}{\partial q_j}\right)=\left\langle\frac{\partial E}{\partial q},\frac{\partial F}{\partial p}\right\rangle-\left\langle\frac{\partial E}{\partial p},\frac{\partial F}{\partial q}\right\rangle$$

which is skew–symmetric, bilinear, and satisfies the Jacobi identity and the Leibniz rule: $(EF,H)=E(F,H)+F(E,H)$. A sequence of smooth functions $\{f_i\}$ is called involutive if $(f_i,f_j)=0$, $\forall i,j$.

In order to prove the integrability of the Bargmann system (3.2) or (3.3), we now introduce a system of smooth functions $\{F_m\}_{m=0}^{\infty}$:

$$F_m = -\tfrac{1}{2}\langle\Lambda^{m+1}p, q\rangle - \tfrac{1}{2}\langle\Lambda^m(p+q), p+q\rangle\langle p, q\rangle - \frac{1}{2}\sum_{j=0}^{m}\begin{vmatrix} \langle\Lambda^j q, q\rangle & \langle\Lambda^j p, q\rangle \\ \langle\Lambda^{m-j}p, q\rangle & \langle\Lambda^{m-j}p, p\rangle \end{vmatrix}. \quad (3.5)$$

We specially point out that $F_0 = H$.

Lemma 3.2. The inner-product

$$\left\langle\frac{\partial F_m}{\partial p}, \frac{\partial F_n}{\partial q}\right\rangle$$

is symmetrical about m and n, i.e.

$$\left\langle\frac{\partial F_m}{\partial p}, \frac{\partial F_n}{\partial q}\right\rangle = \left\langle\frac{\partial F_n}{\partial p}, \frac{\partial F_m}{\partial q}\right\rangle \qquad \forall m, n. \quad (3.6)$$

Proof.

$$\frac{\partial F_m}{\partial p} = -\tfrac{1}{2}\Lambda^{m+1}q - \langle p, q\rangle\Lambda^m q - \langle p+q, q\rangle\Lambda^m p - \tfrac{1}{2}(\langle p, \Lambda^m p\rangle + \langle q, \Lambda^m q\rangle)q$$

$$-\sum_{k=1}^{m}(\langle q, \Lambda^k q\rangle\Lambda^{m-k}p - \langle p, \Lambda^{k-1}q\rangle\Lambda^{m-k+1}q) \quad (3.7)$$

$$\frac{\partial F_n}{\partial p} = -\tfrac{1}{2}\Lambda^{n+1}p - \langle p, q\rangle\Lambda^n p - \langle p+q, p\rangle\Lambda^n q - \tfrac{1}{2}(\langle p, \Lambda^n p\rangle + \langle q, \Lambda^n q\rangle)p$$

$$-\sum_{j=1}^{n}(\langle p, \Lambda^j p\rangle\Lambda^{m-j}q - \langle p, \Lambda^{j-1}p\rangle\Lambda^{m-j+1}p). \quad (3.8)$$

Substituting (3.7) and (3.8) into

$$\left\langle\frac{\partial F_m}{\partial p}, \frac{\partial F_n}{\partial q}\right\rangle$$

and careful calculation, it is not difficult for us to see that

$$\left\langle\frac{\partial F_m}{\partial p}, \frac{\partial F_n}{\partial q}\right\rangle$$

is the sum of a number of items which are symmetrical about m and n. So (3.6) is true. Two identities below are used in the above calculation

$$\sum_{j=1}^{n} (\langle p, \Lambda^j p\rangle\langle q, \Lambda^{n-j}q\rangle - \langle q, \Lambda^j q\rangle\langle p, \Lambda^{n-j}p\rangle) = \langle p, \Lambda^n p\rangle\langle q, q\rangle - \langle q, \Lambda^n q\rangle\langle p, p\rangle$$

$$\sum_{j=1}^{m} (\langle q, \Lambda^j q\rangle\langle p, \Lambda^{m-j}q\rangle - \langle p, \Lambda^j p\rangle\langle q, \Lambda^{m-j}q\rangle) = \langle q, \Lambda^m q\rangle\langle p, p\rangle - \langle p, \Lambda^m p\rangle\langle q, q\rangle.$$

Theorem 3.3.

$$(F_m, F_n) = 0 \qquad \forall m, n \in Z^+. \tag{3.9}$$

Proof.

$$(F_m, F_n) = \left\langle \frac{\partial F_m}{\partial q}, \frac{\partial F_n}{\partial p} \right\rangle - \left\langle \frac{\partial F_m}{\partial p}, \frac{\partial F_n}{\partial q} \right\rangle = \left\langle \frac{\partial F_m}{\partial q}, \frac{\partial F_n}{\partial p} \right\rangle - \left\langle \frac{\partial F_n}{\partial p}, \frac{\partial F_m}{\partial q} \right\rangle = 0.$$

Proposition 3.4. The Hamiltonian systems

$$(F_m):\ q_{t_m} = \partial F_m/\partial p \qquad p_{t_m} = -\partial F_m/\partial q \qquad m = 0, 1, 2, \ldots \tag{3.10}$$

are completely integrable in the Liouville sense. Especially, the system (B) or (3.3) is completely integrable and its involutive system of conserved integrals is $\{F_m\}$.

Theorem 3.5. Let (q, p) be a solution of the Bargmann system (B). Then u and v determined by (3.1) satisfy a stationary nonlinear Levi equation

$$X_N + \alpha_1 X_{N-1} + \cdots \alpha_N X_0 = 0 \tag{3.11}$$

where α_j are decided by $\lambda_1, \ldots \lambda_N$.

Proof. Acting with the operator $(J^{-1}K)^l = \mathscr{L}^l$ upon

$$G_0 = \sum_{j=1}^{N} \nabla\lambda_j$$

and noting (2.4) and (2.6), we have

$$\mathscr{L}^l G_0 = \sum_{j=1}^{N} \lambda_j^l \nabla\lambda_j. \tag{3.12}$$

Consider the polynomial $(\alpha_0 = 1)$

$$p(\lambda) = \prod_{j=1}^{N} (\lambda - \lambda_j) = \alpha_0\lambda^N + \alpha_1\lambda^{N-1} + \cdots + \alpha_N. \tag{3.13}$$

Acting with the operator

$$J \sum_{l=0}^{N} \alpha_{N-l}$$

upon (3.12) and using (2.7) and (3.13), we obtain (3.11).

4. Production of F_m

Theorem 4.1. Let (q,p) be a solution of the Bargmann system (B). Then the nonlinearization of (2.12) (i.e. the time part of the Lax representation for the Levi hierarchy) under (3.1) exactly produces the Hamiltonian system (F_m).

Proof. According to (3.12) and $G_{j-1}=\mathscr{L}^{j-1}G_0$ $(j=1,2,\ldots)$, we have

$$\begin{pmatrix} G_{j-1}^{(1)} \\ G_{j-1}^{(2)} \end{pmatrix} = \begin{pmatrix} \langle p+q, \Lambda^{j-1}p\rangle \\ -\langle p+q, \Lambda^{j-1}q\rangle \end{pmatrix} \qquad j=1,2,\ldots. \tag{4.1}$$

So

$$\begin{cases} G_{j-1}^{(2)} - G_{j-1}^{(1)} = -\langle p+q, \Lambda^{j-1}(p+q)\rangle \\ G_{j-1,x}^{(1)} = \langle p, \Lambda^j p\rangle + \langle p, p+q\rangle\langle p+q, \Lambda^{j-1}(p+q)\rangle \\ G_{j-1,x}^{(2)} = \langle q, \Lambda^j q\rangle + \langle q, p+q\rangle\langle p+q, \Lambda^{j-1}(p+q)\rangle . \end{cases} \tag{4.2}$$

Substituting (4.2) into (2.12), letting $y\to(q+p,p)$, $t\to t_m$, $\lambda\to\Lambda$, $u\to-\langle p+q,q\rangle$, $v\to\langle p+q,p\rangle$, calculating it and noting $G_{-1}=(0,1)^T$, we get

$$\begin{aligned} q_{t_m} = {} & -\tfrac{1}{2}\Lambda^{m+1}q - \tfrac{1}{2}\langle p+q, p+q\rangle\Lambda^m q - \langle p+q, q\rangle\Lambda^m p \\ & + \sum_{j=1}^{m} \left(\tfrac{1}{2}\langle p+q, \Lambda^{j-1}(p+q)\rangle\Lambda^{m+1-j}q - \tfrac{1}{2}(\langle\Lambda^j q, q\rangle + \Lambda^j p, p\rangle)\Lambda^{m-j}q \right. \\ & \left. - \langle\Lambda^j q, q\rangle\Lambda^{m-j}p\right) \\ p_{t_m} = {} & \tfrac{1}{2}\Lambda^{m+1}p + \langle p+q, p\rangle\Lambda^m q + \tfrac{1}{2}\langle p+q, p+q\rangle\Lambda^m p \\ & + \sum_{j=1}^{m} \left(-\tfrac{1}{2}\langle p+q, \Lambda^{j-1}(p+q)\rangle\Lambda^{m+1-j}p + \tfrac{1}{2}(\langle\Lambda^j q, q\rangle + \Lambda^j p, p\rangle)\Lambda^{m-j}p \right. \\ & \left. + \langle\Lambda^j p, p\rangle\Lambda^{m-j}q\right) \end{aligned}$$

i.e.

$$\begin{aligned} q_{t_m} = {} & -\tfrac{1}{2}\Lambda^{m+1}q - \langle p+q, q\rangle\Lambda^m p - \langle p, q\rangle\Lambda^m q - \tfrac{1}{2}(\langle p, \Lambda^m p\rangle + \langle q, \Lambda^m q\rangle)q \\ & - \sum_{j=1}^{m} (\langle q, \Lambda^j q\rangle\Lambda^{m-j}p - \langle p, \Lambda^{j-1}q\rangle\Lambda^{m-j+1}q) = \frac{\partial F_m}{\partial p} \end{aligned} \tag{4.3}$$

$$\begin{aligned} p_{t_m} = {} & \tfrac{1}{2}\Lambda^{m+1}p + \langle p+q, p\rangle\Lambda^m q + \langle p, q\rangle\Lambda^m p + \tfrac{1}{2}(\langle p, \Lambda^m p\rangle + \langle q, \Lambda^m q\rangle)p \\ & + \sum_{j=1}^{m} (\langle p, \Lambda^j p\rangle\Lambda^{m-j}q - \langle p, \Lambda^{j-1}q\rangle\Lambda^{m-j+1}p) = -\frac{\partial F_m}{\partial q}. \end{aligned} \tag{4.4}$$

Thus, the proof of theorem 4.1 is completed.

5. The involutive representation of solutions of the Levi hierarchy

Since the Poisson bracket $(H, F_m)=0$, the Hamiltonian systems (H) and (F_m) are compatible, and their flows $g_0^x, g_m^{t_m}$, which are the solution operators of the corresponding intial-value problems, commute [1].

Define

$$\begin{pmatrix} q(x,t_m) \\ p(x,t_m) \end{pmatrix} = g_0^x g_m^{t_m} \begin{pmatrix} q(0,0) \\ p(0,0) \end{pmatrix}$$

which is called the involutive solution of the consistent equations (B) and (F_m), and is a smooth function of (x, t_m).

Theorem 5.1. Let $(q(x,t_m), p(x,t_m))^T$ be an involutive solution of the consistent system (B) and (F_m). Then

$$u(x,t_m) = -\langle p+q, q\rangle \qquad v = \langle p+q, p\rangle \tag{5.1}$$

satisfy the Levi evolution equation

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = J\mathscr{L}^m G_0 = X_m \qquad m = 0, 1, 2, \ldots. \tag{5.2}$$

Proof. In view of (5.1), (4.4), (4.3) and (3.2), through a series of calculations we have

$$\begin{aligned} \frac{\partial u}{\partial t_m} &= -2\left\langle q, \frac{\partial q}{\partial t_m}\right\rangle - \left\langle q, \frac{\partial p}{\partial t_m}\right\rangle - \left\langle p, \frac{\partial q}{\partial t_m}\right\rangle \\ &= -2\left\langle q, \frac{\partial F_m}{\partial p}\right\rangle + \left\langle q, \frac{\partial F_m}{\partial q}\right\rangle - \left\langle p, \frac{\partial F_m}{\partial p}\right\rangle \\ &= \langle q, \Lambda^{m+1} q\rangle + \langle p+q, q\rangle\langle p+q, \Lambda^m(p+q)\rangle \\ &= (-\langle p+q, \Lambda^m q\rangle)_x \end{aligned} \tag{5.3}$$

$$\begin{aligned} \frac{\partial v}{\partial t_m} &= 2\left\langle p, \frac{\partial p}{\partial t_m}\right\rangle + \left\langle p, \frac{\partial q}{\partial t_m}\right\rangle + \left\langle q, \frac{\partial p}{\partial t_m}\right\rangle \\ &= -2\left\langle p, \frac{\partial F_m}{\partial q}\right\rangle + \left\langle p, \frac{\partial F_m}{\partial p}\right\rangle - \left\langle q, \frac{\partial F_m}{\partial q}\right\rangle \\ &= \langle p, \Lambda^{m+1} p\rangle + \langle p+q, p\rangle\langle p+q, \Lambda^m(p+q)\rangle \\ &= (-\langle p+q, \Lambda^m p\rangle)_x. \end{aligned} \tag{5.4}$$

From (3.12), we obtain

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = J\begin{pmatrix} \langle p+q, \Lambda^m p\rangle \\ -\langle p+q, \Lambda^m q\rangle \end{pmatrix} = J\sum_{j=0}^{N} \lambda_j^m \nabla\lambda_j = J\mathscr{L}^m G_0 = X_m \qquad m = 0, 1, 2, \ldots.$$

Remark 5.2. In the above sense, under the Bargmann constraint (3.1) the spatial and time of the Lax pair for the Levi hierarchy of evolution equations are nonlinearized to be the canonical equation $(F_0) = (H)$, (F_m), respectively. Both of them are completely integrable in the Liouville sense. Furthermore, the solution of the Levi equation (5.2) is reduced to solving the compatible system of two nonlinear ordinary differential equations (3.2), (3.10) and an algebraic operation (5.1).

As a special case of theorem 5.1, choosing $m=1$ and $v=0$ we can obtain the involutive representation of the solution of the well known Burgers equation

$$u_t = u_{xx} + 2uu_x \tag{5.5}$$

Corollary 5.3. Let $(q(x,t), p(x,t))^T$ be the involutive solution of the compatible system $(F_0)=(H), (F_1)$. Let

$$u(x,t) = -\langle p+q, q\rangle \tag{5.6}$$

$$v(x,t) = \langle p+q, p\rangle = 0. \tag{5.7}$$

Then $u(x,t)$ decided by (5.6) satisfies the Burgers equation (5.5), i.e. (5.5) has a kind of the involutive solution $u = -\langle p+q, q\rangle = p, p\rangle - \langle q, p\rangle$.

Proof. As $m=1$ and $v=0$, we have

$$J\mathscr{L}G_0 = X_1 = \begin{pmatrix} -u_{xx}+2uu_x \\ 0 \end{pmatrix}.$$

So, the solution of the Burgers equation (5.5) is reduced to solving the compatible system (F_0), (F_1) which are completely integrable in the Liouville sense and an algebric operation (5.6).

Remark 5.4. On the Neumann constraint produced by

$$G_{-1} = \sum_{j=1}^{N} \nabla\lambda_j$$

and integrability of the corresponding system, a similar result is proved in [26].

Acknowledgments

The author is much indebted to professor Cao Cewen for his help. This project has been supported by the Natural Science Foundation of Educational Committee Liaoning Province, China.

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